Coherent risk measures

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Research interests

- Financial Mathematics, Mathematical Economics,
- Functional Analysis

A general model of risk

$$(\Omega, \mathcal{X}, \rho)$$

- Ω denotes the set of possible future scenarios.
- A financial position is described by a random variable $x : \Omega \to \mathbb{R}$ where $x(\omega)$ is the payoff of the position at the end of the trading period if the scenario $\omega \in \Omega$ is realized. We will denote the space of available financial positions with $\mathcal{X}$.
- A risk measure is a function $\rho : \mathcal{X} \to \mathbb{R}$ that assigns to each $x \in \mathcal{X}$ the value $\rho(x)$. Roughly speaking, $\rho(x)$ represents the money one could potentially lose by investing in x.
- Investments analysts and financial regulators use specific risk measures to determine the risk of a financial position.

Acceptable positions

From the point of view of a financial regulator (e.g. Hellenic Capital Market Commission), $\rho(x)$ is viewed as a capital requirement for the financial institution x. This requirement is put into place to ensure that the institution x will not take on excess leverage and become insolvent. A position set $x \in \mathcal{X}$ is said to be acceptable, whenever $\rho(x) \leq 0$.

$$\mathcal{A} = \{x \in \mathcal{X} \mid \rho(x) \leq 0\}$$

How can we calculate $\rho(x)$?

Value at Risk

Definition

The Value at Risk at level $\lambda \in (0,1)$ of a position $x \in \mathcal{X}$ is given by

$$VaR_\lambda(x) = \inf\{m \mid P[x + m\mathbf{1} < 0] \leq \lambda\}$$

In financial terms, $VaR_\lambda(x)$ is the smallest amount of capital which, if added to x and invested in the risk-free asset $\mathbf{1}$, keeps the probability of a negative outcome below the level λ.

History of *VaR*

- In the late 1980s, *VaR* emerged as a distinct concept in the insurance industry. The triggering event was the stock market crash of 1987.
- In 1994, J. P. Morgan published the methodology and *VaR* had been exposed to the pubic eye for the first time. Since then, *VaR* has been controversial.
- A common complaint among academics is that *VaR* is not subadditive (i.e. $VaR(x+y) \nleq VaR(x) + VaR(y)$)
- Nowadays, *VaR* is still a popular risk measure. Nonetheless, it is criticized by a number of academics and practitioners for its role in the financial crisis of 2007-2008.

Coherent risk measures

In the milestone paper (Coherent measures of risk, P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, **Math. Fin.**, 1999) the authors establish an axiomatic theory of risk measures.

Definition

A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is said to be a coherent risk measure if the following axioms are satisfied:

1. Monotonicity: $x \geq y \Rightarrow \rho(x) \leq \rho(y)$,
2. Positive homogeneity: $\rho(\lambda x) = \lambda \rho(x)\ \forall \lambda \geq 0$,
3. Cash invariance: $\rho(x + m\mathbf{1}) = \rho(x) - m\ \forall m \in \mathbb{R}$,
4. Subadditivity: $\rho(x + y) \leq \rho(x) + \rho(y)\ \forall x, y \in \mathcal{X}$.

Coherent alternatives to VaR_λ

This theory has a significant implication in financial industry. Today several regulators have replaced VaR with alternative risk measures that satisfy the coherence axioms.

Conditional Value at Risk

The Conditional Value at Risk at level $\lambda \in (0,1)$ of a position $x \in X$ is given by

$$CVaR_\lambda(x) = \frac{1}{\lambda}\int_0^\lambda VaR_\gamma(x)d\gamma$$

- How we came up with the above formula? (Functional analysis)
- How we can calculate $CVaR_\lambda(x)$? (Numerical simulation)

Representation of Coherent risk measures

Suppose that $\Omega = \{1, ..., n\}$, then $\mathcal{X} = \mathbb{R}^n$. In this framework a probability measure $P : \Omega \to [0, 1]$ can be represented as a vector $P = (P(1), P(2), ..., P(n))$ where each $P(i)$ denotes the probability of event i. We denote the class of all probability measures with $\mathcal{P}$

Theorem

A mapping $\rho : X \to \mathbb{R}$ is a coherent risk measure if and only if there exists a convex subset $\mathcal{C}$ of $\mathcal{P}$ such that

$$\rho(x) = \sup\{E_P(-x) \mid P \in \mathcal{C}\} = \sup\{-P \cdot x \mid P \in \mathcal{C}\}$$

What about the case where Ω is an infinite set???

In classical mathematical finance it is customary to assume a priori the existence of a probability measure. Nowadays, researchers tend to consider model free markets, without imposing any probabilistic assumption.

In this framework, methods of Banach lattice theory can replace the lack of probabilistic tools. In particular, in this theory probabilistic laws are understood in terms of the order structure of the space.

Definition

A Banach space X equipped with a vector lattice ordering $(X, \geq)$ is said to be a Banach lattice, if for each $x, y \in X$ we have that

$$|x| \leq |y| \Rightarrow ||x|| \leq ||y||, \text{ where } |x| = x \vee (-x)$$

$$L_p(\mu), 1 \leq p \leq \infty, f \leq g \text{ a.e.}$$

Theorem(Biagini-Frittelli, 2009)

Any risk measure $\rho : X \to \mathbb{R}$ on a Banach lattice X is continuous.

Theorem (Fenchel-Moreau)

Let $\phi : X \to (-\infty, \infty]$ be a convex function on a Banach space X. If ϕ is lower semicontinuous, then ϕ admits the following representation.

$$\phi(x) = \sup_{f \in X^*} (\langle f, x \rangle - \phi^*(f)),$$

where $\phi^*(f) = \sup_{x \in X}(\langle f, x \rangle - \phi(x))$

Corollary

Any risk measure ρ on X admits the following representation.

$$\rho(x) = \sup_{f \in (X^*)_+} \{\langle f, -x \rangle - \rho^*(f)\},$$

w^*-dual representation on L_∞

Theorem (Delbaen, 2000)

A proper convex increasing functional $\phi : L_\infty(\mathbb{P}) \to (-\infty, \infty]$ admits the representation
$\phi(x) = \sup_{f \in L_1(\mathbb{P})_+}(\langle f, x \rangle) - \phi^*(f)),$ for any $x \in L_\infty(\mathbb{P})$ iff ϕ satisfies the *Fatou property*: $\phi(x) \leq \liminf \phi(x_n)$ for any bounded sequence (x_n) in $L_\infty(\mathbb{P})$ with $x_n \xrightarrow{a.e.} x$.

What about free-models?

Unbounded order convergence

Definition

In a Banach lattice X, a sequence (x_n) is **order convergent** to $x \in X (x_n \xrightarrow{o} x)$ if there exists another sequence (z_n) such that:

- $z_n \downarrow 0$,
- $|x_n - x| \leq z_n$ for all n

Let (f_n) be a sequence in $L_p(\mu)$, then we have that $f_n \xrightarrow{o} 0$ in L_p iff $f_n \xrightarrow{a.e.} 0$ and there exists $g \in L_p$ such that $|f_n| \leq g$ a.e.

Definition (Nakano, Ann. Math., 1948)

In a Banach lattice X, a sequence (x_n) is **unbounded order convergent** to $x \in X (x_n \xrightarrow{uo} x)$ if $|x_n - x| \wedge y \xrightarrow{o} 0$ for each $y \in X_+$.

Definition

A functional $\phi : X \to (-\infty, \infty]$ is said to be lower σ-unbounded order semi-continuous (σ-uo l.s.c.) if $\phi(x) \leq \liminf \phi(x_n)$ for any norm bounded sequence (x_n) in X with $x_n \xrightarrow{uo} x$.

Theorem (N. Gao, F.X)

Let Y be an order continuous space with weak units and $X = Y^*$. For a proper increasing convex functional $\phi : X \to (-\infty, \infty]$, the following are equivalent.

1. ϕ is w^*-l.s.c.
2. $\phi(x) = \sup_{y \in Y_+}(\langle x, y\rangle - \phi^*(y))$ for any $x \in X$, where $\phi^*(y) = \sup_{x \in X}(\langle x, y\rangle - \phi(x))$ for each $y \in Y$.
3. ϕ is σ-uo l.s.c.

Corollary

Let Φ be an Orlicz function such that $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$. For any proper convex increasing functional $\phi : L_\Phi(\mu) \to (-\infty, \infty]$, the following are equivalent.

1. *ϕ admits the representation*

$$\phi(f) = \sup_{g\in(H_\Psi(\mu))_+} \left(\int_\Omega fg \mathrm{d}\mu - \phi^*(g) \right) \text{ for any } f \in L_\Phi(\mu),$$

   where

$$\phi^*(g) = \sup_{f\in L_\Phi(\mu)} \left(\int_\Omega fg \mathrm{d}\mu - \phi(f) \right) \text{ for each } g \in H_\Psi(\mu).$$

2. *$\phi(f) \leq \liminf \phi(f_n)$ whenever $\sup_n f_{n\Phi} < \infty$ and $f_n \xrightarrow{a.e.} f$.*

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